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ON THE ANALYSIS OF UNILATERALLY SUPPORTED PLATES BY FINITE ELEMENT MODELS

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Abstract—Some variational formulations to the unilaterally supported bent plate problem are considered with regard to finite element, matrix-displacement methods of solution. The fulfillment of the boundary inequalities related to the discrete problem is discussed in accordance to different relaxed formulations.

1. INTRODUCTION

Flexure of thin elastic plates with unilaterally constrained edges has received some attention from various standpoints. The statement of the problem in the context of the theory of variational inequalities is presented in [1]. Existence and uniqueness of the solution in a suitable function space have been discussed by Toscano and Maceri for the plate with elastic (Winkler) support at the boundary and some results of regularity have been given as well[2]. Results on the extent of the loss of contact for a simply supported plate in the vicinity of a corner (i.e. rising from the support) have been deduced by means of an *ad hoc* technique of superposition in a recent work of Keer and Mak[3], with regard to the particular case of a quarter infinite plate submitted to a concentrated load. Moreover, experimental investigations have been performed in the past[4] and recent[5] times.

The relevance of the problem with regard to structural practice is apparent, also from some standard textbooks—see, e.g. ([6], p. 123; [7], Section 5.5.2). From this point of view, the evaluation of deflections and stresses at meaningful points of the plate is of major interest. On this purpose, the use of finite element approximations seems fruitful in the opinion of the authors. On the other hand, numerical methods appear to afford, at the moment, the nearly unique tools of solution—see, e.g. [8]. However, some questions can arise in connection with the algebraic model employed, when unilateral constraints are enforced at the boundary. Namely, if an unknown function (displacement or stress resultant) is represented by means of a nonlinear interpolate, then a unilateral constraint enforced at a finite number of prescribed points[9, 10] (boundary mesh nodes) leads to solve a problem including linear inequalities, but this fact does not imply in principle fulfillment of the constraint everywhere on the boundary of the plate, which comes to behave as unilaterally pointwise constrained. Conversely, if respect of the constraint is pursued everywhere, then the algebraic problem to be solved embodies nonlinear inequalities. On the other hand such a drawback is peculiar also to the numerical management of a number of unilaterally constrained problems in structural engineering—see, e.g. [11].

In this paper, attention is devoted to plate bending in presence of a unilateral, rigid support at the boundary. Indeed, this case of constraint seems not only paradigmatic, but also of a major practical interest. Moreover, an extension to different cases (e.g. unilateral elastic support at the boundary) is possible without substantial differences. Reference is made to matrix-displacement and mixed finite element methods of solution. The essential, unilateral constraint on deflections at the boundary (Section 2) is relaxed by embodying in the current variational formulations of the problem the support reaction as an independent, sign-constrained function. As a consequence, an unconstrained deflection field can be referred to *a priori*, since unilaterality on deflections at the boundary is enforced by making the starting functional stationary (Section 3). In this way it becomes possible to fulfill the constraint on the deflections in a weakened form, on the ground of a suitable, previously assumed description of the support reaction which, in turn, comes to play the role of Lagrange multiplier of the compatibility condition on displacements at the boundary. This aspect is examined with particular regard to the use of linear interpolates for representing the support reaction (Section 4). Some examples end the paper (Section 5).

2. BASIC RELATIONS

The plate is considered as a bidimensional solid which occupies the open, bounded, simply connected domain Ω in the Euclidean space $\mathbb{R}^3$. The boundary $\partial\bar{\Omega}$ of Ω is assumed piecewise smooth, i.e. a corner is admitted at the junction of two consecutive, differentiable arcs. The set of the vertices of $\partial\bar{\Omega}$ is denoted by Z_B, B is the generic element of Z_B. A Cartesian, orthogonal reference system $(0, x_i, z;\ i = 1, 2)$ spans $\mathbb{R}^3$, domain Ω belongs to the $z = 0$ plane, $\partial\bar{\Omega}$ is referred to the curvilinear abscissa b, Fig. 1a). A surface load $p(x_i)$ acts onto Ω, a bending couple $\bar{M}(b)$ and a load $\bar{V}(b)$ (specific values) are distributed along the boundary, p and $\bar{V}$ act in the direction z. The plate is unilaterally supported on $\partial\bar{\Omega}$ and admitted in equilibrium.

The (symmetric) moment tensor $M_{ij}(i, j = 1, 2)$ describes the stress state in the plate. The specific shear force in the direction x_i is $Q_i = \partial_j M_{ij}$†, Fig. 1 b). Specific bending moment M, twisting moment, T, shear force Q transmitted through a linear element ds of Ω are

$$M = M_{ij}n_i n_j, \quad T = M_{ij}n_i t_j, \quad Q = n_j \partial_i M_{ij}, \qquad (1_{1,2,3})$$

where n_i and t_i are components of the unit normal vector $\mathbf{n}$ and tangential vector $\mathbf{t}$ to ds, $\mathbf{t}$ being obtained from $\mathbf{n}$ by a counterclockwise rotation of $\pi/2$. If ds is moved to the boundary $\partial\Omega$, $\mathbf{n}$ becomes the outward unit normal vector and bending moment M and Kirchhoff shear force

$$V = Q + \frac{\mathrm{d}T}{\mathrm{d}b} \qquad (2)$$

are the specific generalized stresses transmitted across $\partial\Omega$. A possible discontinuity of T at a vertex of the boundary must be viewed as the concentrated, transverse force

$$H_B = (T)_{B+0} - (T)_{B-0}. \qquad (3)$$

Stresses and external loads are related by the equations:

$$\partial_{ij}M_{ij} + p = 0 \quad \text{in } \Omega \qquad (4)$$

$$M - \bar{M} = 0 \quad \text{on } \partial\Omega \qquad (5)$$

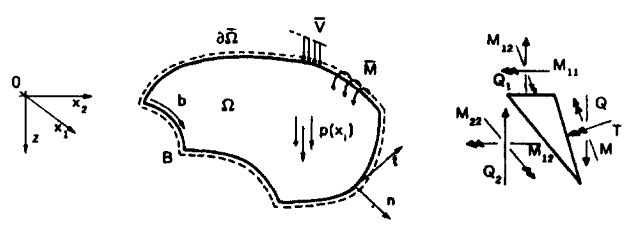

Fig. 1.

†The symbol ∂_i denotes differentiation with respect to x_i, ∂_{ij} stands for $\partial_i\partial_j$. Repeated subscripts mean summation over the range 1, 2. The symbol $\Sigma_{Q \in Z_Q}$ means summation ranging over all the elements Q of the set Z_Q.

and the inequalities

$$V - \bar{V} \leqslant 0 \quad \text{on } \partial\Omega \tag{6$_1$}$$

$$H_B \leqslant 0 \quad \forall B \in Z_B, \tag{6$_2$}$$

as the support can react only in the direction $z \leqslant 0$.

The deflection at a generic point of $\Omega \cup \partial\bar{\Omega}$ is denoted by w. Slope ϑ in the direction of $\mathbf{n}$ and curvature tensor χ_{ij} are defined as

$$\vartheta = n_i \partial_i w, \quad \chi_{ij} = -\partial_{ij} w. \tag{7$_{1,2}$}$$

Besides inequalities (6), the unilateral behaviour of the support requires

$$w \leqslant 0 \quad \text{on } \partial\bar{\Omega} \tag{8}$$

$$\int_{\partial\Omega} (V - \bar{V}) w \, \mathrm{d}b = 0 \quad \text{on } \partial\Omega \tag{9$_1$}$$

$$H_B w_B = 0 \quad \forall B \in Z_B, \tag{9$_2$}$$

where w_B is the deflection at the point B. Equations (9) take place because the support can react only if the plate touches it.

Moment and curvature tensors are related by

$$M_{ij} = C_{ijkl}\chi_{kl}, \quad \chi_{ij} = \bar{C}_{ijkl} M_{kl}, \tag{10$_{1,2}$}$$

where

$$C_{ijkl} = D[(1 - \nu)\delta_{ik}\delta_{jl} + \nu\delta_{ij}\delta_{kl}] \tag{11$_1$}$$

$$\bar{C}_{ijkl} = [(1 + \nu)\delta_{ik}\delta_{jl} - \nu\delta_{ij}\delta_{kl}][(1 - \nu)^2 D]^{-1} \tag{11$_2$}$$

are the (symmetric, positive definite) elastic stiffness and compliance tensors, D is the flexural rigidity of the plate, δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise).

The following bilinear identities (integration by parts) hold, provided that functions M_{ij} and w are sufficiently regular:

$$\int_\Omega M_{ij}\partial_{ij} w \, \mathrm{d}\Omega = -\int_\Omega \partial_i w \partial_j M_{ij} \, \mathrm{d}\Omega + \int_{\partial\Omega} \left(M\vartheta + T\frac{\mathrm{d}w}{\mathrm{d}b} \right) \mathrm{d}b \tag{12$_1$}$$

$$\int_\Omega M_{ij}\partial_{ij} w \, \mathrm{d}\Omega = \int_\Omega w \partial_{ij} M_{ij} \, \mathrm{d}\Omega + \int_{\partial\Omega} \left(M\vartheta - Vw \right) \mathrm{d}b - \sum_{B \in Z_B} H_B w_B. \tag{12$_2$}$$

Equation (12$_2$) follows from (12$_1$) by assuming full regularity for M_{ij}—in particular, continuity and differentiability for twisting moment T with respect to the abscissa b on $\partial\Omega$.

The Reissner's functional related to the problem takes the form:

$$J_R(M_{ij}, w) = -\int_\Omega \left(M_{ij}\partial_{ij} w + \frac{1}{2}\bar{C}_{ijkl} M_{ij} M_{kl} + pw \right) \mathrm{d}\Omega + \int_{\partial\Omega} (\bar{M}\vartheta - \bar{V}w) \, \mathrm{d}b. \tag{13}$$

The moment and deflection fields which solve the plate problem with regard to the operator formulation—equations and inequalities—make functional J_R stationary over the closed, convex set related to the compatibility constraint (8) on the deflection w. The converse holds as well.

3. THE SUPPORT REACTION AS A "SLACK" VARIABLE

Since a reaction plays the role of an external, unknown force which equals inner stresses at the boundary, inequalities (6) are replaced by the relationships:

$$R + \bar{V} - V = 0 \quad R \leqslant 0 \quad \text{on } \partial\Omega \qquad (14_{1,2})$$

$$S_B - H_B = 0 \quad S_B \leqslant 0 \quad \forall B \in Z_B, \qquad (15_{1,2})$$

where $R = R(b)$ denotes the reaction per unit of length along $\partial\Omega$ and S_B the lumped reaction at a point $B \in Z_B$. The analogy between R, S_B and a slack variable of an algebraic inequality is quite apparent. As a consequence, functional $J_R(M_{ij}, w)$ is replaced by the functional

$$\bar{J}_R(M_{ij}, R, \{S_B\}, w, \{w_B\}) = -\int_\Omega \left(M_{ij}\partial_{ij}w + \frac{1}{2}\bar{C}_{ijkl}M_{ij}M_{kl} + pw\right)\mathrm{d}\Omega + \int_{\partial\Omega} \bar{M}\vartheta \,\mathrm{d}b - \int_{\partial\Omega} (R + \bar{V})w \,\mathrm{d}b - \sum_{B\in Z_B} S_B w_B \qquad (16)$$

which differs from J_R, eqn (13), only for the part concerning the boundary. Vectors $\{S_B\}$ and $\{w_B\}$ collect lumped reactions and displacements at the points of Z_B.

Functional $\bar{J}_R$ is substantially equivalent to J_R, but it allows a slightly different constrained variational approach to the problem. On this regard, consider R and $\{S_B\}$ subjected to inequalities (14_2), (15_2) and deflections w no longer as sign-constrained along the boundary of the plate. Hence functional $\bar{J}_R$ is to be viewed as weakly concave with regard to R and $\{S_B\}$ (besides M_{ij}), for deflection w must fulfill in solution constraint (8). As a consequence, the following inequality holds for the variation of $\bar{J}_R$ over $\partial\bar{\Omega}$

$$(\delta\bar{J}_R)_{\partial\bar{\Omega}} = -\int_{\partial\Omega} w\delta R \,\mathrm{d}b - \sum_{B\in Z_B} w_B\delta S_B + \int_{\partial\Omega} (\bar{M} - M)\delta\vartheta \,\mathrm{d}b - \int_{\partial\Omega} (R + \bar{V} - V)\delta w \,\mathrm{d}b - \sum_{B\in Z_B} (S_B - H_B)\delta w_B \leqslant 0. \qquad (17)$$

Stationary conditions of $\bar{J}_R$ over $\partial\bar{\Omega}$ with respect to the (now unconstrained) deflections are obtained by the vanishing of the last two inner products of eqn (17) and they coincide with eqns (14_1), (15_1). By virtue of (14_2), (15_2), variations δR, δS_B must be taken less than zero if R, $\{S_B\}$ are equal to zero, hence stationary conditions with respect to R, $\{S_B\}$ are obtained by enforcing nonnegativity on the first two inner products of (17). This yields

$$w \leqslant 0, \quad \int_\Omega Rw \,\mathrm{d}b = 0 \quad \text{on } \partial\Omega \qquad (18_{1,2})$$

$$w_B \leqslant 0, \quad S_B w_B = 0 \quad \forall B \in Z_B. \qquad (19_{1,2})$$

Therefore, constraints (8), (9) at the boundary are fulfilled by making functional $\bar{J}_R$ stationary under the constraints (14_2), (15_2) on the support reaction.

Suppose now the plate subdivided into contiguous domains by a mesh of lines. All the points $B \in Z_B$ are boundary nodes of the mesh, Z_G denotes the set of nodes lying on $\partial\Omega$, Z_D the set of the inner nodes, G and D are typical elements of Z_G and Z_D respectively. Two domains can have in common at most the points belonging to their boundaries between two consecutive vertices. The eth domain is denoted by Ω^e, its boundary by $\partial\bar{\Omega}^e$. The sides of $\partial\bar{\Omega}^e$ are denoted by $\partial\Omega^e$, the vertices belonging to Ω, $\partial\Omega$, Z_B are still denoted by D, G, B and $Z_D{}^e$, $Z_G{}^e$, $Z_B{}^e$ are the sets of such vertices for the domain Ω^e.

In each domain the properties of regularity required by the relationships of Section 2 for M_{ij} and w are supposed fulfilled, while across the line separating two adjacent domains a discontinuity is admitted either on the moment field or on the slope ϑ in the direction of the normal to the line. Therefore the boundary $\partial\Omega^e$ is split into the parts

$\partial\Omega_b{}^e = \partial\Omega^e \cap \partial\Omega$ and $\partial\Omega_d{}^e = \partial\Omega_M{}^e \cup \partial\Omega_\vartheta{}^e$, a discontinuity for M, or ϑ, being admitted on $\partial\Omega_M{}^e$, or $\partial\Omega_\vartheta{}^e$. Identity ($12_2$) takes now the form

$$\sum_e \left[\int_{\Omega^e} M_{ij}\partial_{ij}w \, d\Omega - \int_{\partial\Omega_\vartheta{}^e} M\vartheta \, db \right] = \sum_e \left[\int_{\Omega^e} w\partial_{ij}M_{ij} \, d\Omega + \int_{\partial\Omega_M{}^e} M\vartheta \, db \right.$$
$$\left. + \int_{\partial\Omega_b{}^e} M\vartheta \, db - \int_{\partial\Omega^e} Vw \, db - \sum_{D\in Z_D{}^e} H_D{}^e w_D - \sum_{G\in Z_G{}^e} H_G{}^e w_G - \sum_{B\in Z_B{}^e} H_B{}^e w_B \right], \tag{20}$$

where $H_\bullet{}^e$ denotes the jump of T, eqn (3), at a vertex of $\partial\bar{\Omega}^e$. Functional J_R, eqn (16), becomes

$$J_{Rd}(M_{ij}, R, \{S_B\}, w, \{w_B\}) = \sum_e \left[- \int_{\Omega^e} \left(M_{ij}\partial_{ij}w + \frac{1}{2} C_{ijkl}M_{ij}M_{kl} + pw \right) d\Omega \right.$$
$$\left. + \int_{\partial\Omega_\vartheta{}^e} M\vartheta \, db + \int_{\partial\Omega_b{}^e} \bar{M}\vartheta \, db + \int_{\partial\Omega_d{}^e} Vw \, db - \int_{\partial\Omega_b{}^e} (R + \bar{V})w \, db \right] - \sum_{B\in Z_B} S_B w_B \tag{21}$$

and its first variation over $\partial\bar{\Omega}$ is

$$(\delta J_{Rd})_{\partial\Omega} = -\sum_e \int_{\partial\Omega_b{}^e} w\delta R \, db - \sum_{B\in Z_B} w_B \delta S_B + \sum_e \left[\int_{\partial\Omega_b{}^e} (\bar{M} - M)\delta\vartheta \, db \right.$$
$$\left. - \int_{\partial\Omega_b{}^e} (R + \bar{V} - V)\delta w \, db \right] - \sum_{B\in Z_B} (S_B - \sum_e H_B{}^e)\delta w_B + \sum_{G\in Z_G} \sum_e H_G{}^e \delta w_G \tag{22}$$

with $(\delta J_{Rd})_{\partial\Omega} \leqslant 0$ for J_{Rd} stationary.

Stationary conditions with respect to reactions are again ($19_{1,2}$) and ($18_{1,2}$) for each $\partial\Omega_b{}^e$. Stationary conditions with respect to deflections over $\partial\bar{\Omega}$—i.e. equilibrium conditions at the boundary—are eqns (5), (14_1) for each $\partial\Omega_b{}^e$ and

$$S_B - \sum_e H_B{}^e = 0 \quad \forall B \in Z_B \tag{23}$$

$$\sum_e H_G{}^e = 0 \quad \forall G \in Z_G. \tag{24}$$

Equation (23) is the extension of eqn (15_2) to the case of more subdomains having a vertex at the same point B. Equation (24) is the statement of continuity for the twisting moment at a point G of $\partial\Omega$. As a consequence, a lumped reaction at a point as G is not required because of the discretization of functional J_R. If the presence of a lumped reaction is enforced, then a specific constraint of nonpositivity on the displacement w_G, to be fulfilled in solution, is carried into the problem. The converse holds as well. Hence, enforcing *a priori* nonpositivity on the deflection at a node as G leads to a description of the reaction (and to an evaluation of the inner stresses at the boundary) in some manner extraneous to the problem, even if the plate can in principle undergo concentrated loads along the boundary by means of a discontinuity on the twisting moment. On the other hand, at a point at G, or B, continuity is not required for shear V along the boundary and, consequently, for reaction R.

The considerations so far exposed apply directly to the particular variational statements at the ground of finite element methods for plate bending. Taking into account eqn (12_1), a functional *à la* Herrmann[12, 13] is obtained from J_{Rd}:

$$J_H(M_{ij}, R, \{S_B\}, w, \{w_B\}) = \sum_e \left[- \int_{\Omega^e} \left(\frac{1}{2} C_{ijkl}M_{ij}M_{kl} - \partial_i w \partial_j M_{ij} + pw \right) d\Omega - \int_{\partial\Omega^e} T \right.$$
$$\left. \times \frac{dw}{db} db - \int_{\partial\Omega_M{}^e} M\vartheta \, db + \int_{\partial\Omega_b{}^e} (\bar{M} - M)\vartheta \, db - \int_{\partial\Omega_b{}^e} (R + \bar{V})w \, db \right] - \sum_{B\in Z_B} S_B w_B. \tag{25}$$

On the other hand, if M_{ij} is selected in the class of moment fields in equilibrium with the load $p(x_i)$, a modified complementary energy functional is obtained:

$$J_C(M_{ij}, R, \{S_B\}, w, \{w_B\}, \{w_G\}, \{w_D\}) = \sum_e \left[\frac{1}{2} \int_{\Omega^e} C_{ijkl} M_{ij} M_{kl} \, d\Omega + \int_{\partial\Omega_f^e} M\vartheta \, db - \int_{\partial\Omega_f^e} Vw \, db - \int_{\partial\Omega_b^e} (\bar{M} - M)\vartheta \, db + \int_{\partial\Omega_b^e} (R + \bar{V} - V)w \, db \right] + \sum_{B \in Z_B} w_B \left(S_B - \sum_e H_B^e \right) - \sum_{G \in Z_G} w_G \sum_e H_G^e - \sum_{D \in Z_D} w_D \sum_e H_D^e. \quad (26)$$

Finally, a modified potential energy functional can be derived from J_{Rd} by assuming in advance full compatibility for w onto Ω:

$$J_P(R, \{S_B\}, w) = \sum_e \left[\int_{\Omega^e} \left(\frac{1}{2} C_{ijkl} \partial_{ij} w \partial_{kl} w \right) d\Omega + \int_{\partial\Omega_b^e} \bar{M}\vartheta \, db - \int_{\partial\Omega_b^e} (R + \bar{V}) w \, db \right] - \sum_{B \in Z_B} S_B w_B. \quad (27)$$

Functionals (25)–(27) lead to saddle-point variational statements under the constraint of nonpositivity for R and $\{S_B\}$. Their stationary conditions with respect to the support reactions are again those exposed for functional J_{Rd}, which are indeed not fulfilled *a priori* in passing from J_{Rd} to J_H, J_C or J_P. When such conditions apply to functional J_P, Kirchhoff shear $V(b)$ and inner forces H must be viewed as functions of w, via the relationships of Section 2. Functionals (26) and (27) are the analogue of functionals employed to develop equilibrium[14], hybrid (assumed stress)[15, 16], compatible[17] finite element models for plate bending, leading to matrix-displacement methods of solution[18].

4. SOME CONSEQUENCES

The outlined scheme allows to provide for an autonomous description of $R(b)$, suitable for a numerical approach. It is apparent in this outlook that identification among the support reaction $R(b)$ and Kirchhoff shear at the boundary of the plate can be in principle achieved only in terms of resultants. On this regard, consider a subdomain Ω^e with a side, $\partial\Omega_b^e$, belonging to $\partial\Omega$. Moreover, represent $R(b)$ onto $\partial\Omega_b^e$ by means of the m-degree complete algebraic polynomial

$$R^e(b) = \boldsymbol{\beta}^T \mathbf{a}$$

where $\boldsymbol{\beta}$, $\mathbf{a}$ are $(m+1)$-vectors, the jth component of $\boldsymbol{\beta}$ is $\beta_j = b^{j-1}$ and $\mathbf{a}$ is the coefficient vector. Owing to a continuous field of deflections $w^e(b)$ onto $\partial\Omega_b^e$, the virtual work performed by $R^e(b)$ takes the form:

$$\int_{\partial\Omega_b^e} R^e(b) w^e(b) \, db = \int_{\partial\Omega_b^e} \boldsymbol{\beta}^T \mathbf{a} w^e(b) \, db = \mathbf{u}^{e^T} \mathbf{a}, \quad (28)$$

where $\mathbf{u}^e$ is the $(m+1)$-vector whose ith component is

$$u_j^e = \int_{\partial\Omega_b^e} b^{j-1} w^e(b) \, db. \quad (29)$$

Therefore, such a modelling of $R(b)$ leads to take into account, *sub specie* of work, the deflections at the boundary of the plate as the set of average values (29), where the monomial b^{j-1} is the weight. It is more convenient, in practice, to interpolate $R(b)$ over

$\partial\Omega_b{}^e$ by means of the Lagrange polynomial of degree m [19]:

$$R^e(b) = \mathbf{\Phi}^T(b)\rho^e, \tag{30}$$

where $\mathbf{\Phi}(b)$ is the $(m + 1)$-vector of shape functions and ρ^e is the unknown vector of nodal parameters, which are the values taken by R^e over a set of $m + 1$ points (knots), fixed in advance onto $\partial\Omega_b{}^e$. Expression (28) becomes

$$\int_{\partial\Omega_b{}^e} R^e(b)w^e(b)\,\mathrm{d}b = \int_{\partial\Omega_b{}^e} \mathbf{\Phi}^T(b)\rho^e(b)w^e(b)\,\mathrm{d}b = \mathbf{w}^{e^T}\rho^e, \tag{28'}$$

where the jth component of $\mathbf{w}^e$ is

$$w_j^e = \int_{\partial\Omega_b{}^e} \Phi_j(b)w^e(b)\,\mathrm{d}b \tag{29'}$$

and $\Phi_j(b)$ is the jth shape function. In this case, w_j^e is a linear combination of components of $\mathbf{u}$. Constraint (14_2) takes the form

$$R^e(b) = \mathbf{\Phi}^T(b)\rho^e \leqslant 0 \tag{31}$$

and the integral under the first summation of (22) becomes

$$\int_{\partial\Omega_b{}^e} w\delta R\,\mathrm{d}b = \int_{\partial\Omega_b{}^e} w^e\delta[\mathbf{\Phi}^T(b)\rho^e]\,\mathrm{d}b = \int_{\partial\Omega_b{}^e} w^e[\mathbf{\Phi}^T(b)\delta\rho^e]\,\mathrm{d}b. \tag{32}$$

As the whole term in the square brackets at the r.h.s. of (32) must be less than zero if inequality (31) is fulfilled as an equation, stationary for functionals (21), (25)... (27) with respect to R implies again $w^e \leqslant 0$ and vanishing of the inner product (28′), if inequality (31) is required to be fulfilled punctually. It should be observed that this fact requires, in principle, to solve a nonconvex optimum problem for $R(b)$ over $\partial\Omega_b{}^e$, in order to express the position of the extremum points of R^e in dependence of ρ^e and enforce condition (31) at such points. In this way the l.h.s. of (31) becomes a nonlinear function of ρ, moreover inequality (8) must be applied punctually over $\partial\Omega_b{}^e$.

If inequality (31) is replaced by the inequality

$$\rho^e \leqslant 0, \tag{33}$$

then eqn (32) shows that stationary with respect to R implies simply

$$\mathbf{w}^e = \int_{\partial\Omega_b{}^e} \mathbf{\Phi}(b)w^e\,\mathrm{d}b \leqslant 0 \tag{34}$$

$$\rho^{e^T}\mathbf{w}^e = 0$$

and both inequalities (33) and (34) are linear. This approach (see, e.g. [20], in a different context) leads to weaken condition (18_1), but also to relax condition (14_2). Namely the first

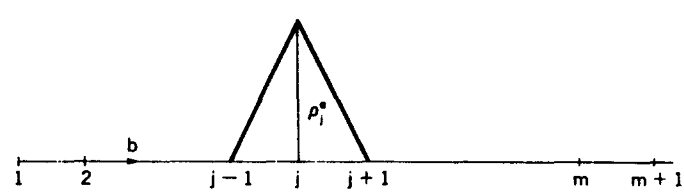

Fig. 2.

one is enforced only in the average sense previously exposed, the second on the set of the knots only.

As, after all, $R(b)$ is required to weak condition (8), a more coherent, but by no means less useful representation of $R(b)$ is achievable by adopting a piecewise linear Lagrange interpolation with $m+1$ knots, including the end points of $\partial\Omega_b^e$. Indeed, in this case conditions (31) and (33) become equivalent. Moreover, quite elementary expressions are obtained for w_j^e. Namely, assume for the sake of simplicity $\partial\Omega_b^e$ rectilinear, Fig. 2. Then it follows:

$$\Phi_j(b) = \left[\frac{1}{2} + \frac{1}{l_{j-1,j}}\left(b - b_{j-1} - \frac{l_{j-1,j}}{2}\right)\right] + \left[\frac{1}{2} + \frac{1}{l_{j,j-1}}\left(\frac{l_{j,j+1}}{2} - b + b_j\right)\right]$$

$$w_j^e = \left(\bar{w}_{j-1,j}^e - \frac{l_{j-1,j}}{6}\bar{\varphi}_{j-1,j}^e\right)\frac{l_{j-1,j}}{2} + \left(w_{j,j+1}^e + \frac{l_{j,j+1}}{6}\bar{\varphi}_{j,j+1}^e\right)\frac{l_{j,j+1}}{2},$$

where b_j is the abscissa of the jth knot and

$$\bar{w}_{\cdot\cdot}^e = \frac{1}{b_{\cdots}}\int_{l_{\cdots}} w^e(b)\,\mathrm{d}b, \quad \bar{\varphi}_{\cdot\cdot}^e = \frac{12}{l_{\cdots}}\int_{l_{\cdots}}\left(\frac{l_{\cdots}}{2} - b\right)w^e(b)\,\mathrm{d}b$$

are recognizable as the average displacement and slope over the subinterval among two consecutive knots.

The above considerations are independent of the representation adopted for $w(x_i)$, which can be taken according to finite element models already developed for plate bending in the previously exposed variational contexts. On this regard, assume the deflection $w^e(b)$ as represented by a n-degree complete polynomial over $\partial\Omega_b^e$. Then $w^e(b)$ depends on $n+1$ unknown parameters and w_j^e, eqn (29′), becomes a linear expression of these parameters. As a consequence, inequality (34) becomes a system of $m+1$ inequalities in $n+1$ variables and the rank of the coefficient matrix of this system is equal to min $\{m+1, n+1\}$, since shape functions $\mathbf{\Phi}(b)$ are a basis for $\partial\Omega_b^e$. It is just worth noting that the same coefficient matrix, transpose and sign-changed, affects vector ρ^e in the equilibrium equations derived from the fourth integral of (22). The algebraic problem deriving from the whole discretization is a linear complementarity problem in the set of variables $\mathbf{w}^e$ and ρ^e[21]. If the number, $m+1$, of the knots in the interpolation of $R^e(b)$ is equal to $n+1$, then

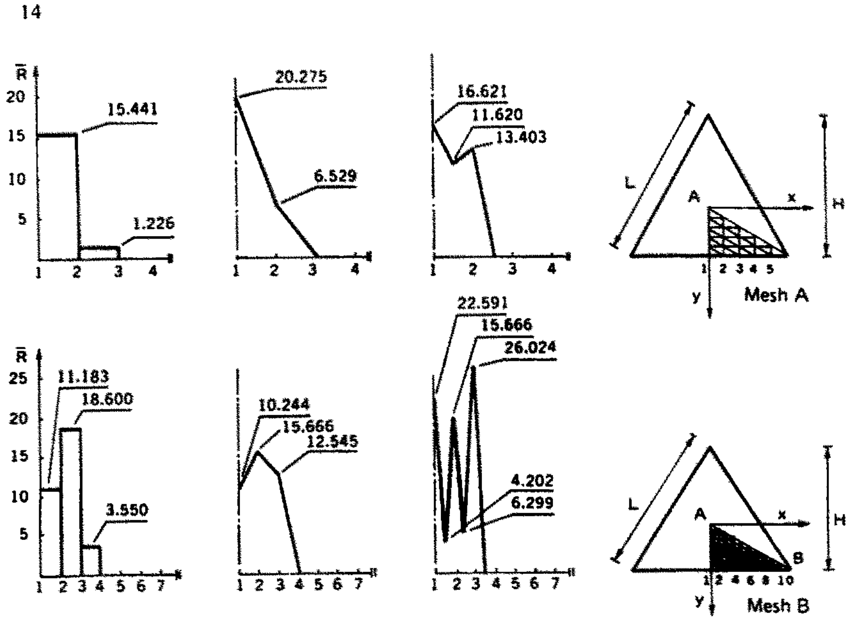

Fig. 3. Functions $\bar{R}(b) = R(b)(L/P)$ for HCT[17] element, concentrated load P.

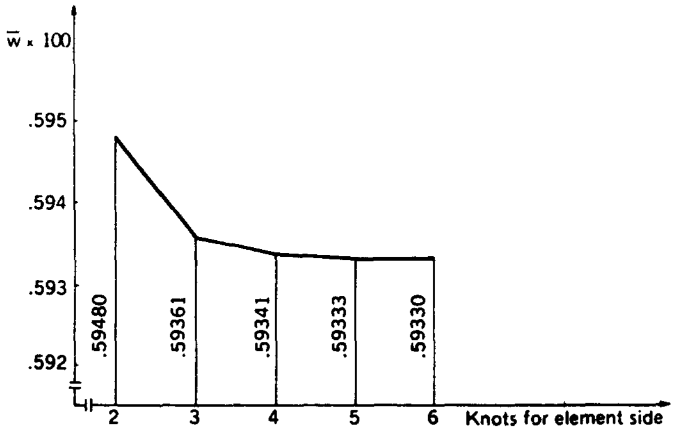

Fig. 4. Central deflection $\bar{w} = w/(PH^2/D)$, concentrated load P, mesh A.

inequality (34) actually constraints all the parameters of $w^e(b)$. A more accurate control on the deflections on $\partial\Omega_b^e$ could be reached if a greater number of knots is employed ($m > n$) to interpolate $R(b)$. Nevertheless, only $n + 1$ inequalities can, at most, result really active†. After all, the description of the behaviour of the plate, even along the boundary, comes to depend substantially on the interpolation model taken in advance for the deflection field.

5. AN APPLICATION

A triangular equilateral plate of uniform thickness and elastic constants $E = 206010$ MPa, $\nu = 0.3$ is considered. The plate is assumed unilaterally supported along the boundary and submitted alternatively to a concentrated load P at the center and to a uniform load q. The HCT finite element developed in Ref. [17] (12 d.o.f compatible

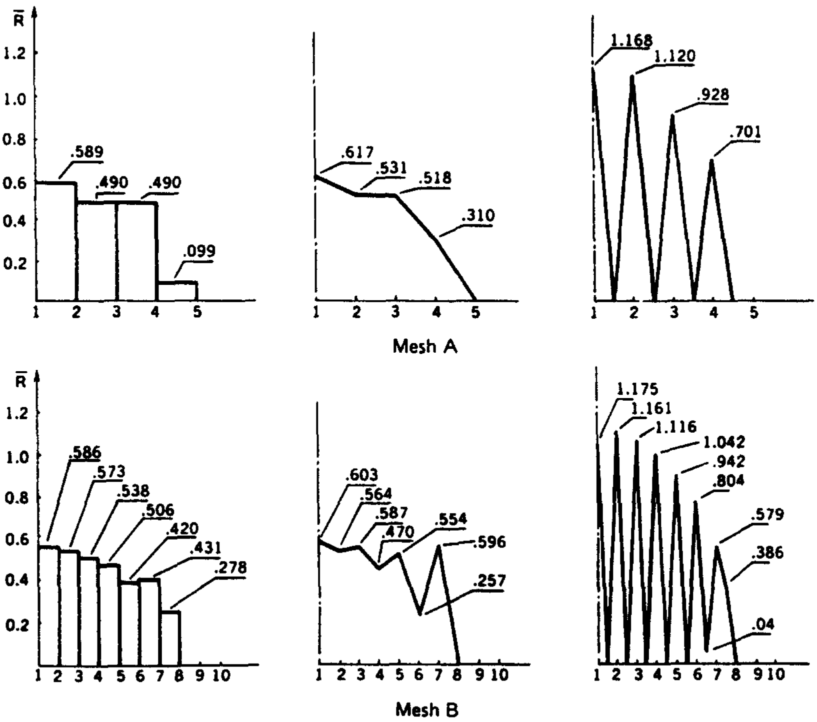

Fig. 5. Functions $\bar{R}(b) = R(b)(2 \cdot q/H)$ for HCT 12[17] element, uniform load q.

†It is quite transparent the analogy between R and a plastic activation rate, w and the distance of the stress point from the yield locus in a plasticity problem with linearized associate flow laws[22].

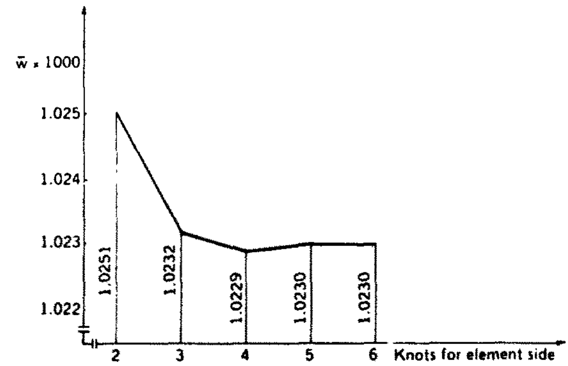

Fig. 6. Central deflection $\bar{w} = w/(qH^4/D)$, uniform load q, mesh A.

model) is adopted. Regular meshes obtained by subdividing the plate side in ten (mesh A) and twenty (mesh B) equal subintervals are employed, Fig. 3. Numerical developments have been performed on the system CDC CYBER 76 of the CINECA (Casalecchio di Reno, Bologna).

Concentrated load

The upper part of Fig. 3 depicts respectively the cases of $R(b)$ constant, linear, piecewise linear with a central knot for each element side, mesh A. The lower part of Fig. 3 is the analogue for mesh B. In the (piecewise) linear interpolation model, $R(b)$ is assumed continuous at the mesh nodes. The deflections at the plate center, for mesh A and $2 \div 6$ knots piecewise linear interpolates (constant interval) for $R(b)$, are reported in Fig. 4. The central deflection evaluated by assuming $R(b)$ constant for each element side is 0.0059480 PH^2/D, whereas the value of 0.0059328 PH^2/D is obtained by assuming only the boundary mesh nodes as unilaterally supported (element sides are free), mesh A. The deflections w_A at the plate center and w_B at the corner for mesh B are reported in the two first columns of Table 1 (†). The values obtained by means of the 13 d.o.f. equilibrium model (EQT [14]) are also reported. In this case the unilateral constraint has been enforced only on the average side deflection.

Uniform load

Figure 5 represents $R(b)$ as in the previous loading case, for the meshes A and B. The central deflection for mesh A and different numbers of knots are reported in Fig. 6, which is analogous to Fig. 4. For the same mesh, the central deflection evaluated for $R(b)$ constant for each element boundary side is 0.00102429 qH^4/D and the deflection for the case of only mesh nodes unilaterally supported is 0.00102219 qH^4/D. The rightmost part of Table 1 collects the results for the deflections w_A and w_B and the moment M at the plate center.

Table 1. Deflections and moment at the plate center for a central load P and a uniformly distributed load q

	$w_A \times 10^{-2}$	$w_B \times 10^{-2}$	$w_A \times 10^{-2}$	$w_B \times 10^{-2}$	$M \times 10^{-1}$
0	0.59953	0.32063	0.10310	0.00653	0.24202
1	0.59979	0.32077	0.10314	0.00684	0.24208
2	0.59969	0.32436	0.10306	0.00611	0.24194
3	0.59955	0.32155	0.10311	0.00652	0.24202
EQT[14]	0.60356	0.32196	0.10340	0.00724	0.24188
Bilateral support [6, art. 72]	0.575	—	0.10288	—	0.24074
Multiplier	PH^2/D	PH^2/D	qH^4/D	qH^4/D	qH^2

†As the values for more than 3 knots are substantially coincident with the value obtained for 3 knots for side, they are not reported.

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